An Introduction to Logical Relations by (Re-)Inventing the Tait Method

Following Harper (2025)

Presented by Yanning Chen @ ProSE Seminar

STLC

$$\frac{x \in \Gamma}{\Gamma \vdash x : A}\textsc{Var} \quad \frac{}{\Gamma \vdash \text{yes} : \text{ans}}\textsc{Yes} \quad \frac{}{\Gamma \vdash \text{no} : \text{ans}}\textsc{No} \quad \frac{}{\Gamma \vdash \langle\rangle : 1}\textsc{Unit}$$

$$\frac{\Gamma \vdash M_1 : A \quad \Gamma \vdash M_2 : B}{\Gamma \vdash (M_1, M_2) : A \times B}\textsc{Prod} \quad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash M.1 : A}\textsc{Prj1} \quad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash M.2 : B}\textsc{Prj2}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}\textsc{Abs} \quad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}\textsc{App}$$

(Also, denote head β-reduction as $M \mapsto N$)

Type Safety of STLC

Theorem (Termination of STLC):
If $\emptyset \vdash M : \mathrm{ans}$, then either $M \overset{*}{\mapsto} \mathrm{yes}$ or $M \overset{*}{\mapsto} \mathrm{no}$.

Lemma (Progress of STLC):
If $\emptyset \vdash M : A$, then either $\mathbf{value}\ M$ or $\exists N, M \mapsto N$.

Lemma (Preservation of STLC):
If $\emptyset \vdash M : A$ and $M \mapsto N$, then $\emptyset \vdash N : A$.

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Type Safety of STLC

Theorem (Termination of STLC):
If $\emptyset \vdash M : \mathrm{ans}$, then either $M \overset{*}{\mapsto} \mathrm{yes}$ or $M \overset{*}{\mapsto} \mathrm{no}$.

Syntactic Approach: Progress + Preservation, but not accounting for non-divergence!

Semantic Approach: direct proof (via logical relations)

Termination, attempt 1

Theorem (Termination of STLC):
If $\emptyset \vdash M : \mathrm{ans}$, then either $M \overset{*}{\mapsto} \mathrm{yes}$ or $M \overset{*}{\mapsto} \mathrm{no}$.

Proof:

Termination, attempt 1

Theorem (Termination of STLC):
If $\emptyset \vdash M : \mathrm{ans}$, then either $M \overset{*}{\mapsto} \mathrm{yes}$ or $M \overset{*}{\mapsto} \mathrm{no}$.

Proof:
By induction on the derivation of $\emptyset \vdash M : \mathrm{ans}$.

- Case $\textsc{Yes}$ and $\textsc{No}$: trivial.

Termination, attempt 1

Theorem (Termination of STLC):
If $\emptyset \vdash M : \mathrm{ans}$, then either $M \overset{*}{\mapsto} \mathrm{yes}$ or $M \overset{*}{\mapsto} \mathrm{no}$.

Proof:
By induction on the derivation of $\emptyset \vdash M : \mathrm{ans}$.

- Case $\textsc{Yes}$ and $\textsc{No}$: trivial.
- Case $\textsc{Lft}$:
  IH: if $\mathrm{ans} = \mathrm{ans} \times B$ and $\emptyset \vdash M : \mathrm{ans}$, then either $M \overset{*}{\mapsto} \mathrm{yes}$ or $M \overset{*}{\mapsto} \mathrm{no}$. ???

Termination, attempt 2 (generalizing the type)

Theorem (Termination of STLC):
If $\emptyset \vdash M : A$, then $\exists N$, $M \overset{*}{\mapsto} N$ and $\mathbf{value}\ N$, where

$$\frac{}{\mathbf{value}\ \mathrm{yes}}\textsc{Yes} \qquad \frac{}{\mathbf{value}\ \mathrm{no}}\textsc{No} \qquad \frac{}{\mathbf{value}\ \langle\rangle}\textsc{Unit}$$

$$\frac{}{\mathbf{value}\ \langle M_1, M_2 \rangle}\textsc{Prod} \qquad \frac{}{\mathbf{value}\ \lambda x.M}\textsc{Lam}$$

Proof:

Termination, attempt 2 (generalizing the type)

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If $\emptyset \vdash M : A$, then $\exists N$, $M \overset{*}{\mapsto} N$ and $\mathbf{value}\ N$, where

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Proof:

- Case $\textsc{Lft}$:
  Assume: $\emptyset \vdash M : A \times B$
  IH: $\exists N$, $M \overset{*}{\mapsto} N$ and $\mathbf{value}\ N$.
  wts. $\exists N^?$ s.t. $M.1 \overset{*}{\mapsto} N^?$ and $\mathbf{value}\ N^?$.

Termination, attempt 2 (generalizing the type)

Theorem (Termination of STLC):
If $\emptyset \vdash M : A$, then $\exists N$, $M \overset{*}{\mapsto} N$ and $\mathbf{value}\ N$, where

$$\frac{}{\mathbf{value}\ \mathrm{yes}}\textsc{Yes} \quad \frac{}{\mathbf{value}\ \mathrm{no}}\textsc{No} \quad \frac{}{\mathbf{value}\ \langle\rangle}\textsc{Unit}$$

$$\frac{}{\mathbf{value}\ \langle M_1, M_2\rangle}\textsc{Prod} \quad \frac{}{\mathbf{value}\ \lambda x.M}\textsc{Lam}$$

Proof:

- Case $\textsc{Lft}$:
  $\exists N_1 N_2$, $M \overset{*}{\mapsto} \langle N_1, N_2\rangle$ (by IH, preservation and $\mathbf{value}$)
  wts. $M.1 \overset{*}{\mapsto} N_1$ and $\mathbf{value}\ N_1$. ???

Termination, attempt 3 (strengthening $\mathbf{value}$)

Conjecture (Termination of STLC?):
If $\emptyset \vdash M : A$, then $\exists N$, $M \overset{*}{\mapsto} N$ and 👍$\mathbf{value}\ N$, where

$$\frac{}{👍\mathbf{value}\ \mathrm{yes}}\textsc{Yes} \qquad \frac{}{👍\mathbf{value}\ \mathrm{no}}\textsc{No} \qquad \frac{}{👍\mathbf{value}\ \langle\rangle}\textsc{Unit}$$

$$\frac{👍\mathbf{value}\ M_1 \qquad 👍\mathbf{value}\ M_2}{👍\mathbf{value}\ \langle M_1, M_2\rangle}\textsc{Prod} \qquad \frac{?}{👍\mathbf{value}\ \lambda x.M}\textsc{Lam}$$

Termination, attempt 3 (strengthening $\mathbf{value}$)

Conjecture (Termination of STLC?):
If $\emptyset \vdash M : A$, then $\exists N$, $M \overset{*}{\mapsto} N$ and 👍$\mathbf{value}\ N$

- Consider $\dfrac{👍\mathbf{value}\ M_1 \quad 👍\mathbf{value}\ M_2}{👍\mathbf{value}\ \langle M_1, M_2 \rangle}\textsc{Prod}$

  ? What about $\langle \langle Y, N \rangle .1, \ldots \rangle$

Termination, attempt 3 (strengthening $\mathbf{value}$)

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  ? What about $\langle\langle Y, N\rangle.1, \ldots\rangle$
- Consider $\dfrac{?}{👍\mathbf{value}\ \lambda x.M}\textsc{Lam}$

  How to fill the hole?

$$\frac{👍\mathbf{value}\ M[N/x]}{👍\mathbf{value}\ \lambda x.M}\textsc{Lam}? \quad \frac{M[N/x] \overset{*}{\mapsto} M' \quad 👍\mathbf{value}\ N \Rightarrow 👍\mathbf{value}\ M'}{👍\mathbf{value}\ \lambda x.M}\textsc{Lam}?$$

A better 👍value

Theorem (Termination of STLC):
If $\emptyset \vdash M : A$, then $\exists N$, $M \overset{*}{\mapsto} N$ and 👍$\mathbf{value}\ N$

$$\frac{}{\text{👍}\mathbf{value}\ \mathrm{yes}}\textsc{Yes} \qquad \frac{}{\text{👍}\mathbf{value}\ \mathrm{no}}\textsc{No} \qquad \frac{}{\text{👍}\mathbf{value}\ \langle\rangle}\textsc{Unit}$$

$$\frac{M_1 \overset{*}{\mapsto} M_1' \quad M_2 \overset{*}{\mapsto} M_2' \quad \text{👍}\mathbf{value}\ M_1' \quad \text{👍}\mathbf{value}\ M_2'}{\text{👍}\mathbf{value}\ \langle M_1, M_2\rangle}\textsc{Prod}$$

$$\frac{M[N/x] \overset{*}{\mapsto} M' \quad \text{👍}\mathbf{value}\ N \Rightarrow \text{👍}\mathbf{value}\ M'}{\text{👍}\mathbf{value}\ \lambda x.M}\textsc{Lam}$$

Note: $\Rightarrow$ means *meta-level implication*.

👍T: $\mapsto$ + 👍value

Theorem (👍 Termination of STLC):
If $\emptyset \vdash M : A$, then 👍$\mathbf{T}\ M$.

$$\frac{M \overset{*}{\mapsto} \text{yes}}{👍\mathbf{T}\ M}\,\textsc{Yes} \qquad \frac{M \mapsto * \text{no}}{👍\mathbf{T}\ M}\,\textsc{No} \qquad \frac{M \overset{*}{\mapsto} \langle\rangle}{👍\mathbf{T}\ M}\,\textsc{Unit}$$

$$\frac{M \overset{*}{\mapsto} \langle M_1, M_2\rangle \quad 👍\mathbf{T}\ M_1 \quad 👍\mathbf{T}\ M_2}{👍\mathbf{T}\ M}\,\textsc{Prod}$$

$$\frac{M \overset{*}{\mapsto} \lambda x.M' \quad 👍\mathbf{T}\ N \Rightarrow 👍\mathbf{T}\ M'[N/x]}{👍\mathbf{T}\ M}\,\textsc{Lam}$$

👍T: $\mapsto$ + 👍value

$$\frac{M \overset{*}{\mapsto} \text{yes}}{👍\mathbf{T}\ M}\textsc{Yes} \qquad \frac{M \mapsto * \text{no}}{👍\mathbf{T}\ M}\textsc{No} \qquad \frac{M \overset{*}{\mapsto} \langle\rangle}{👍\mathbf{T}\ M}\textsc{Unit}$$

$$\frac{M \overset{*}{\mapsto} \langle M_1, M_2\rangle \quad 👍\mathbf{T}\ M_1 \quad 👍\mathbf{T}\ M_2}{👍\mathbf{T}\ M}\textsc{Prod}$$

$$\frac{M \overset{*}{\mapsto} \lambda x.M' \quad 👍\mathbf{T}\ N \Rightarrow 👍\mathbf{T}\ M'[N/x]}{👍\mathbf{T}\ M}\textsc{Lam}$$

Problem: 👍**T** is undecidable and hard to reason!

If only we can know the *intended canonical form* of a term in advance...

Hereditary Termination ($\mathbf{HT}$: type-indexed 👍T)

Conjecture (?Hereditary Termination of STLC):
If $\emptyset \vdash M : A$, then $\mathbf{HT}_A M$.

$$\frac{M \overset{*}{\mapsto} \text{yes}}{\mathbf{HT}_{\text{ans}}(M)}\textsc{Yes} \qquad \frac{M \mapsto * \text{no}}{\mathbf{HT}_{\text{ans}}(M)}\textsc{No} \qquad \frac{M \overset{*}{\mapsto} \langle\rangle}{\mathbf{HT}_1(M)}\textsc{Unit}$$

$$\frac{M \overset{*}{\mapsto} \langle M_1, M_2\rangle \quad \mathbf{HT}_A(M_1) \quad \mathbf{HT}_B(M_2)}{\mathbf{HT}_{A\times B}(M)}\textsc{Prod}$$

$$\frac{M \overset{*}{\mapsto} \lambda x.M' \quad \mathbf{HT}_A(N) \Rightarrow \mathbf{HT}_B(M'[N/x])}{\mathbf{HT}_{A\to B}(M)}\textsc{Lam}$$

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Conjecture (?Hereditary Termination of STLC):
If $\emptyset \vdash M : A$, then $\mathbf{HT}_A M$.

$$\mathbf{HT}_{\text{ans}}(M) := M \overset{*}{\mapsto} \text{yes or } M \overset{*}{\mapsto} \text{no} \quad \mathbf{HT}_1(M) := M \overset{*}{\mapsto} \langle\rangle$$

$$\mathbf{HT}_{A\times B}(M) := M \overset{*}{\mapsto} \langle M_1, M_2\rangle \text{ and } \mathbf{HT}_A(M_1) \text{ and } \mathbf{HT}_B(M_2)$$

$$\mathbf{HT}_{A\to B}(M) := M \overset{*}{\mapsto} \lambda x.M' \text{ and } \mathbf{HT}_A(N) \Rightarrow \mathbf{HT}_B(M'[N/x])$$

Termination, attempt 4 (plain HT)

Conjecture (?Hereditary Termination of STLC):
If $\emptyset \vdash M : A$, then $\mathbf{HT}_A(M)$.

Proof:

- Case $\textsc{Lft}$:

  Assume $\emptyset \vdash M : A \times B$, by IH $\mathbf{HT}_{A \times B}(M)$
  wts. $\mathbf{HT}_A(M.1)$

Termination, attempt 4 (plain HT)

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  Assume $\emptyset \vdash M : A \times B$, by IH $\mathbf{HT}_{A \times B}(M)$
  wts. $\mathbf{HT}_A(M.1)$

  By $\mathbf{HT}_{A \times B}(M)$, we know $M \overset{*}{\mapsto} \langle M_1, M_2 \rangle$ and $\mathbf{HT}_A(M_1)$,
  and observe that $M.1 \overset{*}{\mapsto} \langle M_1, M_2 \rangle.1 \mapsto M_1$.

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If $\emptyset \vdash M : A$, then $\mathbf{HT}_A(M)$.

Proof:

- Case LFT:

  Assume $\emptyset \vdash M : A \times B$, by IH $\mathbf{HT}_{A\times B}(M)$
  wts. $\mathbf{HT}_A(M.1)$

  By $\mathbf{HT}_{A\times B}(M)$, we know $M \overset{*}{\mapsto} \langle M_1, M_2\rangle$ and $\mathbf{HT}_A(M_1)$,
  and observe that $M.1 \overset{*}{\mapsto} \langle M_1, M_2\rangle.1 \mapsto M_1$.

  💡 It suffices to show that $\mathbf{HT}$ is closed under "reverse execution".

Head Expansion a.k.a. “reverse execution”

Lemma (Head Expansion):
If $M \overset{*}{\mapsto} N$ and $\mathbf{HT}_A(N)$, then $\mathbf{HT}_A(M)$.

Proof: by definition of $\mathbf{HT}$.

$$\mathbf{HT}_{\text{ans}}(M) := M \overset{*}{\mapsto} \text{yes or } M \overset{*}{\mapsto} \text{no} \quad \mathbf{HT}_1(M) := M \overset{*}{\mapsto} \langle\rangle$$

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Conjecture (?Hereditary Termination of STLC):
If $\emptyset \vdash M : A$, then $\mathbf{HT}_A(M)$.

Proof:

- Case $\textsc{Lam}$:

  IH2: if $\emptyset = \emptyset, x : A$ and $\emptyset, x : A \vdash M : B$, then $\mathbf{HT}_B(M)$
  wts. $\mathbf{HT}_{A \to B}(\lambda x.M)$???

Termination, attempt 4 (plain HT)

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If $\emptyset \vdash M : A$, then $\mathbf{HT}_A(M)$.

Proof:

- Case $\textsc{Lam}$:

  IH2: if $\emptyset = \emptyset, x : A$ and $\emptyset, x : A \vdash M : B$, then $\mathbf{HT}_B(M)$
  wts. $\mathbf{HT}_{A \to B}(\lambda x.M)$???

  !! Need to generalize over Γ, but $\mathbf{HT}$ applies only to closed terms!

How to deal with open terms?

Given subst γ from variables to terms,
we say $\Gamma' \vdash \gamma : \Gamma$ iff $\forall x : A \in \Gamma, \Gamma' \vdash \gamma(x) : A$.

Subst Lemma:

$$\frac{\Gamma' \vdash \gamma : \Gamma \quad \Gamma \vdash M : A}{\Gamma' \vdash M[\gamma] : A}$$

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Subst Lemma (specialized):

$$\frac{\emptyset \vdash \gamma : \Gamma \quad \Gamma \vdash M : A}{\emptyset \vdash M[\gamma] : A}$$

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Given subst γ from variables to terms,
we say $\Gamma' \vdash \gamma : \Gamma$ iff $\forall x : A \in \Gamma, \Gamma' \vdash \gamma(x) : A$.

Idea: adopting *subst lemma* to $\mathbf{HT}$,

$$\frac{\mathbf{HT}_{\Gamma}(\gamma) \quad \Gamma \gg M \in A}{\mathbf{HT}_{A}(M[\gamma])}$$

where $\mathbf{HT}_{\Gamma}(\gamma) := \forall x : A \in \Gamma, \mathbf{HT}_{A}(x[\gamma])$

How to deal with open terms?

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Idea: adopting *subst lemma* to $\mathbf{HT}$,

$$\Gamma \gg M \in A := \mathbf{HT}_\Gamma(\gamma) \Rightarrow \mathbf{HT}_A(M[\gamma])$$

where $\mathbf{HT}_\Gamma(\gamma) := \forall x : A \in \Gamma, \mathbf{HT}_A(x[\gamma])$

Hereditary Termination, finally

Theorem (FTLR of $\mathbf{HT}$, or Hereditary Termination of STLC):
If $\Gamma \vdash M : A$, then $\Gamma \gg M \in A$. (where $\Gamma \gg M \in A := \mathbf{HT}_\Gamma(\gamma) \Rightarrow \mathbf{HT}_A(M[\gamma])$)

Proof:

- Case $\textsc{Var}$ ($\Gamma \gg \alpha \in A$):
  By assumption, $\alpha : A \in \Gamma$. Assume $\mathbf{HT}_\Gamma(\gamma)$, wts. $\mathbf{HT}_A(\alpha[\gamma])$.
  So $\gamma(\alpha) = M$ s.t. $\mathbf{HT}_A(M)$. But, $\alpha[\gamma] = \gamma(\alpha)$. We are done.

Hereditary Termination, finally

Theorem (FTLR of $\mathbf{HT}$, or Hereditary Termination of STLC):
If $\Gamma \vdash M : A$, then $\Gamma \gg M \in A$. (where $\Gamma \gg M \in A := \mathbf{HT}_\Gamma(\gamma) \Rightarrow \mathbf{HT}_A(M[\gamma])$)

Proof:

- Case Lam ($\Gamma \gg \lambda x.M : A \to B$):
  Assume $\mathbf{HT}_\Gamma(\gamma)$, wts. $\mathbf{HT}_{A\to B}(\lambda x.M[\gamma])$. By definition of $\mathbf{HT}_{A\to B}$, assume $\mathbf{HT}_A(N)$, wts. $\mathbf{HT}_B(M[\gamma][N/x])$.
  By I.H., $\forall \gamma', \mathbf{HT}_{\Gamma,x:A}(\gamma')$ implies $\mathbf{HT}_B(M[\gamma'])$. Specializing I.H. by $\gamma' := \gamma, x \to N$, and note that $\mathbf{HT}_{\Gamma,x:A}(\gamma')$, we have $\mathbf{HT}_B(M[\gamma, x \to N])$, and $\mathbf{HT}_B(M[\gamma][N/x])$.

  Idea: apply γ from premises (by I.H.) to the conclusion.

From Termination to (Weak) β-Normalizing

Termination: *head* β-reduction of well-typed *closed terms* stops at canonical forms ($\mathbf{value}$).

Normalization: *full* β-reduction of well-typed *open terms* stops at β-normal forms (can not step anymore).

$$\frac{M_2 \underset{\beta}{\to} M_2'}{M_1\,M_2 \underset{\beta}{\to} M_1\,M_2'}\textsc{App2} \qquad \frac{M \underset{\beta}{\to} M'}{\lambda x.M \underset{\beta}{\to} \lambda x.M'}\textsc{Abs}$$

$$\frac{M_1 \underset{\beta}{\to} M_1'}{\langle M_1, M_2\rangle \underset{\beta}{\to} \langle M_1', M_2\rangle}\textsc{ProdL} \qquad \frac{M_2 \underset{\beta}{\to} M_2'}{\langle M_1, M_2\rangle \underset{\beta}{\to} \langle M_1, M_2'\rangle}\textsc{ProdR}$$

Normalization of STLC, formally

Normalization: *full* β-reduction of well-typed *open terms* stops at β-normal forms (can not step anymore).

Formally, define $\mathbf{norm}_\beta(M) := \exists N, M \overset{*}{\underset{\beta}{\to}} N$ and $N \underset{\beta}{\not\to}$

Theorem (Normalization of STLC):
If $\Gamma \vdash M : A$, then $\mathbf{norm}_\beta(M)$.

by proving the following lemma

From HT to HN: Kripke LR

$\mathbf{HT}_A(M)$ only applies to *closed terms*, while $\mathbf{HN}$ must deal with *open terms*.

Solution (Kripke LR): index over free variables (Δ), or *“possible worlds”*.

$\mathbf{HN}_A^{\Delta}(M)$ is indexed by variable contexts Δ and types A on well-formed terms $\Delta \vdash M : A$.

Hereditary Normalizing (HN)

$$\mathbf{HT}_{\text{ans}}(M) := M \overset{*}{\mapsto} \text{yes or } M \overset{*}{\mapsto} \text{no} \quad \mathbf{HT}_1(M) := M \overset{*}{\mapsto} \langle\rangle$$

$$\mathbf{HT}_{A\times B}(M) := M \overset{*}{\mapsto} \langle M_1, M_2\rangle \text{ and } \mathbf{HT}_A(M_1) \text{ and } \mathbf{HT}_B(M_2)$$

$$\mathbf{HT}_{A\to B}(M) := M \overset{*}{\mapsto} \lambda x.M' \text{ and } \mathbf{HT}_A(N) \Rightarrow \mathbf{HT}_B(M'[N/x])$$

$$\mathbf{HN}^{\Delta}_{\text{ans}}(M) := \mathbf{norm}_\beta(M) \quad \mathbf{HN}^{\Delta}_1(M) := \mathbf{norm}_\beta(M)$$

$$\mathbf{HN}^{\Delta}_{A\times B}(M) := \mathbf{HN}^{\Delta}_A(M.1) \text{ and } \mathbf{HN}^{\Delta}_B(M.2)$$

$$\mathbf{HN}^{\Delta}_{A\to B}(M) := \forall \Delta' \leq \Delta, \mathbf{HN}^{\Delta'}_A(N) \Rightarrow \mathbf{HN}^{\Delta'}_B(M\,N)$$

HN vs HT

- $\mathbf{HN}^{\Delta}_{A\times B}(M) := \mathbf{HN}^{\Delta}_{A}(M.1)$ and $\mathbf{HN}^{\Delta}_{B}(M.2)$ vs $\mathbf{HT}_{A\times B}(M) := M \overset{*}{\mapsto} \langle M_1, M_2\rangle$ and $\mathbf{HT}_A(M_1)$ and $\mathbf{HT}_B(M_2)$

  M might be a variable, so it probably won't reduce to canonical form. Define $\mathbf{HN}$ via elimination instead of introduction.

$$\frac{\Gamma \vdash M_1 : A \quad \Gamma \vdash M_2 : B}{\Gamma \vdash (M_1, M_2) : A \times B}\textsc{Prod}$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash M.1 : A}\textsc{Prj1} \qquad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash M.2 : B}\textsc{Prj2}$$

HN vs HT

- $\mathbf{HN}^{\Delta}_{A\to B}(M) := \forall \Delta' \leq \Delta, \mathbf{HN}^{\Delta'}_{A}(N) \Rightarrow \mathbf{HN}^{\Delta'}_{B}(MN)$

  $\Delta' \leq \Delta := \forall x, \Delta \vdash x : A \Rightarrow \Delta' \vdash x : A$, i.e. Δ' is an extension of Δ.

  Intuitively: a function can be applied in a larger context by weakening lemma.

  Lemma (Anti-Monotonicity):
  If $\mathbf{HN}^{\Delta(M)}_{A}$ and $\Delta' \leq \Delta$, then $\mathbf{HN}^{\Delta'}_{A}(M)$.

Head Expansion for $\mathbf{HN}$

Lemma (Head Expansion):

If $M \overset{*}{\mapsto} N$ and $\mathbf{HN}^{\Delta}_{A}(N)$, then $\mathbf{HN}^{\Delta}_{A}(M)$.

Proof: 👋.

💡 It suffices to show that $\mathbf{HN}$ is closed under *head expansion* instead of *full expansion*.

FTLR of $\mathbf{HN}$

Theorem (FTLR of $\mathbf{HN}$, or Hereditary Normalizing of STLC):

If $\Gamma \vdash M : A$, then $\forall \Delta, \mathbf{HN}_{\Gamma}^{\Delta}(\gamma) \Rightarrow \mathbf{HN}_{A}^{\Delta}(M[\gamma])$.

Proof: 👋.

One Missing Step: From Hereditary-$\mathcal{P}$ to $\mathcal{P}$

Theorem (FTLR of $\mathbf{HT}$)

If $\Gamma \vdash M : A$, then $\Gamma \gg M \in A$. (where $\Gamma \gg M \in A := \mathbf{HT}_\Gamma(\gamma) \Rightarrow \mathbf{HT}_A(M[\gamma])$)

Theorem (Termination of STLC)

If $\emptyset \vdash M : \mathrm{ans}$, then either $M \overset{*}{\mapsto} \mathrm{yes}$ or $M \overset{*}{\mapsto} \mathrm{no}$.

One Missing Step: From Hereditary-$\mathcal{P}$ to $\mathcal{P}$

Theorem (FTLR of $\mathbf{HT}$)
If $\Gamma \vdash M : A$, then $\Gamma \gg M \in A$. (where $\Gamma \gg M \in A := \mathbf{HT}_\Gamma(\gamma) \Rightarrow \mathbf{HT}_A(M[\gamma])$)

Theorem (Termination of STLC)
If $\emptyset \vdash M : \text{ans}$, then either $M \overset{*}{\mapsto} \text{yes}$ or $M \overset{*}{\mapsto} \text{no}$.

Proof:
Instantiating FTLR with $\Gamma = \emptyset$ and $A = \text{ans}$, $\mathbf{HT}_\emptyset(\gamma) \Rightarrow \mathbf{HT}_A(M[\gamma])$
$\mathbf{HT}_\emptyset(\gamma)$ holds trivially, and $M[\gamma] = M$ because M is closed, so $\mathbf{HT}_A(M)$.

Now we are done by the definition of $\mathbf{HT}$.
$\square$

From HN to Normalizing?

Not so easy!

$\mathbf{HN}^{\Delta}_{A\times B}(M) := \mathbf{HN}^{\Delta}_{A}(M.1)$ and $\mathbf{HN}^{\Delta}_{B}(M.2)$ vs $\mathbf{HT}_{A\times B}(M) := M \overset{*}{\mapsto} \langle M_1, M_2\rangle$ and $\mathbf{HT}_{A}(M_1)$ and $\mathbf{HT}_{B}(M_2)$

1. $\mathbf{HT}$ works on closed term, thus the precondition is trivial, while $\mathbf{HN}$ is not.
2. $\mathbf{HT}$ is defined by introduction, which means we have direct information about M itself, while $\mathbf{HN}$ is defined by elim form like $M.1$.

From $\mathbf{HN}$ to Normalizing?

Theorem (FTLR of $\mathbf{HN}$)
If $\Gamma \vdash M : A$, then $\forall \Delta, \mathbf{HN}^{\Delta}_{\Gamma}(\gamma) \Rightarrow \mathbf{HN}^{\Delta}_{A}(M[\gamma])$.

Theorem (Normalization of STLC)
If $\Gamma \vdash M : A$, then $\mathbf{norm}_{\beta}(M)$.

From $\mathbf{HN}$ to Normalizing?

Theorem (FTLR of $\mathbf{HN}$)
If $\Gamma \vdash M : A$, then $\forall \Delta, \mathbf{HN}_{\Gamma}^{\Delta}(\gamma) \Rightarrow \mathbf{HN}_{A}^{\Delta}(M[\gamma])$.

Theorem (Normalization of STLC)
If $\Gamma \vdash M : A$, then $\mathbf{norm}_{\beta}(M)$.

Proof:
It suffices to show that

1. $\mathbf{HN}_{\Gamma}^{\Gamma}(\iota)$, where $\iota(x) = x$ (i.e., $\Gamma \vdash \iota : \Gamma$)
2. (Adaquacy) If $\Gamma \vdash M : A$ and $\mathbf{HN}_{A}^{\Delta}(M)$, then $\mathbf{norm}_{\beta}(M)$

$\mathbf{HN}_{\Gamma}^{\Gamma}(\iota)$

Theorem: $\forall x : A \in \Gamma, \mathbf{HN}_{A}^{\Gamma}(x)$
(every variable in Γ is hereditarily normalizing at its claimed type).

Proof:
By case analysis on A.

- Case $\textsc{Ans}$:
  Assume $\alpha : \mathrm{ans} \in \Gamma$, wts. $\mathbf{HN}_{\mathrm{ans}}^{\Gamma}(\alpha)$, which is to show $\mathbf{norm}_{\beta}(\alpha)$, and it follows directly from the definition of $\mathbf{norm}_{\beta}$.

$\mathbf{HN}_{\Gamma}^{\Gamma}(\iota)$

Theorem: $\forall x : A \in \Gamma, \mathbf{HN}_{A}^{\Gamma}(x)$
(every variable in Γ is hereditarily normalizing at its claimed type).

Proof:
By case analysis on A.

- Case $\textsc{Prod}$:
  Assume $\alpha : A \times B \in \Gamma$, wts. $\mathbf{HN}_{A\times B}^{\Gamma}(\alpha)$. It suffices to show $\mathbf{HN}_{A}^{\Gamma}(\alpha.1)$ and $\mathbf{HN}_{B}^{\Gamma}(\alpha.2)$.
- Case $\textsc{Lam}$ $(\mathbf{HN}_{A\to B}^{\Gamma}(\alpha))$:
  Assume $\forall\Gamma' \leq \Gamma, \mathbf{HN}_{A}^{\Gamma'}(M)$. It suffices to show that $\mathbf{HN}_{B}^{\Gamma'}(\alpha M)$.
  Applying adaquacy to assumption, we have $\mathbf{norm}_{\beta}(M)$.

So, to prove $\mathrm{HN}_{\Gamma}^{\Gamma}(\iota)$

We want to show that
$\mathbf{HN}_{A}^{\Gamma}(\alpha.1)$, $\mathbf{HN}_{A}^{\Gamma}(\alpha.2)$, and $\mathbf{norm}_{\beta}(M) \Rightarrow \mathbf{HN}_{A}^{\Gamma'}(\alpha M)$

So, to prove $\mathrm{HN}^{\Gamma}_{\Gamma}(\iota)$

We want to show that
$\mathbf{HN}^{\Gamma}_{A}(\alpha.1)$, $\mathbf{HN}^{\Gamma}_{A}(\alpha.2)$, and $\mathbf{norm}_{\beta}(M) \Rightarrow \mathbf{HN}^{\Gamma'}_{A}(\alpha M)$

Generalize it a bit, we define *neutral term* $U := x \mid U.1 \mid U.2 \mid UM$ as terms that stuck regarding head reduction.

And we can define *normalizable neutral term* $\mathbf{nnorm}_{\beta}$:

$$\mathbf{nnorm}_{\beta}(x) := \top$$
$$\mathbf{nnorm}_{\beta}(U.1) := \mathbf{nnorm}_{\beta}(U) \quad \mathbf{nnorm}_{\beta}(U.2) := \mathbf{nnorm}_{\beta}(U)$$
$$\mathbf{nnorm}_{\beta}(UM) := \mathbf{nnorm}_{\beta}(U) \text{ and } \mathbf{norm}_{\beta}(M)$$

And we prove that if $\mathbf{nnorm}_{\beta}(U)$, then $\mathbf{HN}^{\Delta}_{A}(U)$

Pas-de-deux, or the Dance of $\mathbf{norm}_\beta$ and $\mathbf{HN}$

Lemma (Pas-de-deux): $\forall A$ and $\Delta \vdash U, M : A$,

1. If $\mathbf{nnorm}_\beta(U)$, then $\mathbf{HN}^\Delta_A(U)$
2. If $\mathbf{HN}^\Delta_A(M)$, then $\mathbf{norm}_\beta(M)$

Proof: By induction on A,

- Case Lam:
  1. ($\mathbf{HN}^\Delta_{A\to B}(U)$) Assume $\mathbf{nnorm}_\beta(U)$. Let $\Delta' \leq \Delta$, it suffices to show that $\mathbf{HN}^{\Delta'}_A(N)$ implies $\mathbf{HN}^{\Delta'}_B(UN)$. By induction (2), $\mathbf{norm}_\beta(N)$, thus $\mathbf{nnorm}_\beta(UN)$. By induction (1), $\mathbf{HN}^{\Delta'}_B(UN)$.
  2. ($\mathbf{norm}_\beta(M)$) Assume $\mathbf{HN}^\Delta_{A\to B}(M)$. Let $\Delta' := \Delta, x : A \leq \Delta$, we have $\mathbf{HN}^{\Delta'}_B(Mx)$, and by induction (2), $\mathbf{norm}_\beta(Mx)$. By definition, $\mathbf{nnorm}_\beta(x)$, so by induction (1), $\mathbf{HN}^{\Delta'}_A(x)$. Then by analysis on β-reduction, $\mathbf{norm}_\beta(M)$.

From $\mathbf{HN}$ to Normalizing

Theorem (Normalization of STLC)
If $\Gamma \vdash M : A$, then $\mathbf{norm}_\beta(M)$.

Proof:
It suffices to show that

1. $\mathbf{HN}_\Gamma^\Gamma(\iota)$
2. (Adaquacy) If $\Gamma \vdash M : A$ and $\mathbf{HN}_A^\Delta(M)$, then $\mathbf{norm}_\beta(M)$

Both follows from the *pas-de-deux lemma*.

Logical Relation, generalized on $\mathcal{P}$

Conjecture If $\Gamma \vdash M : A$, then $\mathcal{P}_A^\Gamma(M)$.

Proof: Define LR *hereditarily* $\mathcal{P}$ as $h\mathcal{P}_A^\Delta(M)$.

$h\mathcal{P}_1^{\Delta(M)} := M \overset{*}{\mapsto} \langle\rangle$ or $M \overset{*}{\mapsto} U$ and $n\mathcal{P}_1^{\Delta(U)}$
$h\mathcal{P}_{A\times B}^\Delta(M) := h\mathcal{P}_A^\Delta(M.1)$ and $h\mathcal{P}_B^\Delta(M.2)$
$h\mathcal{P}_{A\to B}^\Delta(M) := \forall \Delta' \leq \Delta$, if $h\mathcal{P}_A^{\Delta'}(N)$, then $h\mathcal{P}_B^{\Delta'}(MN)$

Where $n\mathcal{P}$ (*neutrally* $\mathcal{P}$) requires that the argument terms of U to be $h\mathcal{P}$:

$n\mathcal{P}_A^{\Delta,\alpha:A}(\alpha) := \top$
$n\mathcal{P}_A^\Delta(U.1) := n\mathcal{P}_{A\times B}^\Delta(U)$
$n\mathcal{P}_B^\Delta(U.2) := n\mathcal{P}_{A\times B}^\Delta(U)$
$n\mathcal{P}_B^\Delta(UM) := n\mathcal{P}_{A\to B}^\Delta(U)$ and $h\mathcal{P}_A^\Delta(M)$

Reduction Property $\mathcal{P}$

By concluding from our previous proof of *FTLR* and *pas-de-deux*, both hold for $\mathcal{P}$ if:

1. $P_1^{\Delta}(\langle\rangle)$.
2. If $n\mathcal{P}_1^{\Delta}(U)$, then $\mathcal{P}_1^{\Delta}(U)$.
3. If $\mathcal{P}_A^{\Delta}(M.1)$ and $\mathcal{P}_B^{\Delta}(M.2)$, then $\mathcal{P}_{A\times B}^{\Delta}(M)$.
4. If $\mathcal{P}_B^{\Delta,x:A}(Mx)$, then $\mathcal{P}_{A\to B}^{\Delta}(M)$.

We call this kind of property *reduction property*. So,

Theorem (Principle of *reduction property*)
Given *reduction property* $\mathcal{P}$, if $\Gamma \vdash M : A$, then $\mathcal{P}_A^{\Gamma}(M)$.

Conclusion

- To prove termination of STLC: logical relation $\mathbf{HT}$.
- To prove normalization of STLC: Kripke-style LR $\mathbf{HN}$.
- From *FTLR* of $\mathbf{HN}$ to normalization: *pas-de-deux*.
- LR as a general principle: *reduction property*.

Interesting applications:

1. *Strong normalizing* is a *reduction property*.
2. Verify safety of ill-typed programs (think *RustBelt*).
3. LR indexed by source types but on target terms (think *FFI*).

Related Material

- *Harper, Robert. "How to (Re)Invent Tait's Method".* [Link]
  Define *Hereditary Termination* $\mathbf{HT}$
- *Harper, Robert. "Kripke-Style Logical Relations for Normalization".* [Link]
  Define *Hereditary Normalizing* $\mathbf{HN}$
- *Harper, Robert. "Strong Normalization as Transfinite Induction on Reduction".* [Link]
  Generalized LR, and an old but general approach to prove β*-confluence* via *transfinite* $\to$*-induction* (property that satisfies *head expansion*).
- *Harper, Robert. "How to (Re)Invent Girard's Method".* [Link]
  LR for *System F*.